(2). Let the coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) .... (x_n, y_n, z_n), of n points be given or measured, and let $\delta_1, \delta_2, \ldots \delta_n$, be the normals respectively from the n points to any line or surface. Then the probability that a point, taken at random on any of the surfaces of the spheres whose centers are (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_u, z_n), and radii, $\delta_1, \delta_2, \ldots \delta_n$, shall be at the foot of one of these normals is $\dfrac{n}{\pi m(\delta_1^2+\delta_2^2+\ldots+\delta_n^2)}$. And the probability that n points, taken at random on the surfaces of the spheres, shall be at the intersection of these normals with the line or surface is

$$\frac{n^n}{\pi^n m^n(\delta_1^2+\delta_2^2+\ldots+\delta_n^2)^n},$$

which probability is greatest when $\delta_1^2+\delta_2^2+\ldots+\delta_n^2 =$ a minimum.

That is, from (1) and (2), the point, line or surface, which n points make the most probable, is the point line or surface which makes the sum of the squares of the normals upon it, or sum of the squares of the errors of situation a minimum.

SOLUTIONS OF PROBLEMS IN NUMBER EIVE.

Solutions of problems in number five have been received as follows:

From Marcus Baker, 177, 178 and 179; Prof. W. P. Casey, 177 and 178; G. M. Day, 178; Prof. H. T. Eddy, 179; Edgar Frisby, 178; Newton Fitz, 175; Henry Gunder, 175, 177, 178, 179 and 180; Henry Heaton, 175, 177, 178, 179 and 180; Geo. Lilley, 179; Christine Ladd, 177; Prof. H. T. J. Ludwick, 178; Prof. D. J. Mc. Adam, 177, 178 and 179; Prof. Orson Pratt, 176; Werner Stille, 179; E. B. Seitz, 175, 177, 178, 179, 180, 181; Prof. J. Scheffer, 175, 177, 178, 179, 180; Prof. D. Trowbridge, 179.

---

175. "Find the roots of the equation $x^4+Ax^3+Bx^2+Cx+C^2\div A^2=0$."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Multiplying the equation by $4A^2$, then adding $(A^4+8AC-4A^2B)x^2$ to both members, we get

$$4A^2x^4+4A^3x^3+(A^4+8AC)x^2+4A^2Cx+4C^2=(A^4+8AC-4AB)x^2.$$

Extracting the square root, we have

$$2Ax^2+A^2x+2C=\pm x\sqrt{(A^4+8AC-4A^2B)},$$ whence

$$x=\{-A^2\pm\sqrt{(A^4+8AC-4A^2B)}\pm A\sqrt{[2A^2-4B\mp2\sqrt{(A^4+8AC-4A^2B)}]}\}\div4A.$$

176. "Given $u = F\ (y)$, and $y = F'\,[z+xv\,f\ (y)]$;
also $v = F_1(\,t)$, and $t = F_2[z+xv\,f_1(\,t)]$,
to expand (by the differential calculus) u in a series of ascending, positive and integral powers of x. (x not being a function of z.)"

SOLUTION BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.

We have $u = F\ \{\ F'\,[z+xv\,f(y)]\ \} = F''[z+xv\,f(y)]$;
also $f(y) = f\ \{\ F'\,[z+xv\,f(y)]\ \} = f'\ [z+xv\,f(y)]$.
We also have $v = F_1\{\ F_2[z+xvf_1(t)]\ \} = F_3[z+xvf_1(t)]$;
and $f_1\,(t) = f_1\{\ F_2[z+xvf_1(t)]\ \} = f_2[z+xvf_1(t)]$.
Let $p = F''(z)$; $q = f'(z)$; $r = F_3(z)$; $s = f_2(z)$. Then, by Laplace's theorem, we have

$$u = p + \frac{qdp}{dz}.\frac{xv}{1} + d\ \frac{q^2\,dp}{dz^2}.\frac{x^2v^2}{1.2} + d^2\frac{q^3dp}{dz^3}.\frac{x^3v^3}{1.2.3} + \&c., \tag{1}$$

$$v = r + \frac{sdr}{dz}.\frac{xv}{1} + d\ \frac{s^2\,dr}{dz^2}.\frac{x^2v^2}{1.2} + d^2\frac{s^3dr}{dz^3}.\frac{x^3v^3}{1.2.3} + \&c. \tag{2}$$

By squaring (2), cubing, &c., we have

$$v^2 = r^2 + \frac{sd(r^2)}{dz}.\frac{xv}{1} + d\ \frac{s^2d(r^2)}{dz^2}.\frac{x^2v^2}{1.2} + d^2\frac{s^3d(r^2)}{dz^3}.\frac{x^3v^3}{1.2.3} + \&c., \tag{3}$$

$$v^3 = r^3 + \frac{sd(r^3)}{dz}.\frac{xv}{1} + d\ \frac{s^2d(r^3)}{dz^2}.\frac{x^2v^2}{1.2} + d^2\frac{s^3d(r^3)}{dz^3}.\frac{x^3v^3}{1.2.3} + \&c., \tag{4}$$

$$v^4 = r^4 + \frac{sd(r^4)}{dz}.\frac{xv}{1} + d\ \frac{s^2d(r^4)}{dz^2}.\frac{x^2v^2}{1.2} + d^2\frac{s^3d(r^4)}{dz^3}.\frac{x^3v^3}{1.2.3} + \&c., \tag{5}$$

. . . .

Eliminate v, v^2, &c., from (1), by substituting their values as determined in (2), (3), &c. In these substituted values again substitute for v, v^2, &c. Continue the process until v and its powers are excluded; after which add the coefficients of the like powers of x, and reduce their sums, and the result will be

$$\begin{aligned} u &= p \\ &+ \frac{x}{1} \cdot \frac{qdp}{dz}.\ r \\ &+ \frac{x^2}{1.2} \cdot \left(d\ \frac{q^2dp}{dz^2}.r^2 + \frac{q\ dp}{dz}.\frac{sd(r^2)}{dz}\right) \\ &+ \frac{x^3}{1.2.3} \cdot \left(d^2\frac{q^3dp}{dz^3}.r^3 + 2d\ \frac{q^2dp}{dz^2}.\frac{sd(r^3)}{dz} + \frac{q\ dp}{dz}.d\frac{s^2d(r^3)}{dz^2}\right) \end{aligned}$$

$$+\frac{x^4}{1.2.3.4}\cdot\left(d^3\frac{q^4dp}{dz^4}.r^4+3d^2\frac{q^3dp}{dz^3}.\frac{sd(r^4)}{dz}+3d\frac{q^2dp}{dz^2}.d\frac{s^2d(r^4)}{dz^2}\right.$$
$$\left.+\frac{qdp}{dz}.d^2\frac{s^3d(r^4)}{dz^3}\right).$$

For the mth power of x, we have

$$\frac{x^m}{1.2...m}\left(d^{m-1}\frac{q^mdp}{dz^m}.r^m+(m-1)d^{m-2}\frac{q^{m-1}dp}{dz^{m-1}}.\frac{sd(r^m)}{dz}+\frac{(m-1)(m-2)}{1\,.\,2}\right.$$
$$\left.\times d^{m-3}\frac{q^{m-2}dp}{dz^{m-2}}.d\frac{s^2d(r^m)}{dz^2}+\ldots\ldots+\frac{qdp}{dz}.d^{m-2}\frac{s^{m-1}d(r^m)}{dz^{m-1}}\right).$$

By restoring the values of p, q, r and s, we have

$$u=F''(z)$$
$$+\frac{x}{1}\cdot\frac{f'(z).dF''(z)}{dz}.F_3(z)$$
$$+\frac{x^2}{1.2}\cdot\left(d\frac{[f'(z)]^2.dF''(z)}{dz^2}.[F_3(z)]^2+\frac{f'(z).dF''(z)}{dz}.\frac{f_2(z).d[F_3(z)]^2}{dz}\right)$$
$$+\frac{x^3}{1.2.3}\cdot\Big(\&c.$$

This general theorem may be very much condensed, and clearly expressed in the following form:

$$u=F''(z)+A^{(d^0)}\frac{x}{1}+A^{(d^1)}\frac{x^2}{1.2}+A^{(d^2)}\frac{x^3}{1.2.3}+\&c. \qquad \text{(A)}$$

In this, $A^{(d^1)}$ is equal to the binomial differential coefficient of $\frac{x^2}{1.2}$; while $A^{(d^2)}$, $A^{(d^3)}$, &c., represent the second, third, &c., differential expansions of the binomial $A^{(d^1)}$. By reference to the general term, it will be seen that when $m=1$, all the terms of the coefficient vanish, excepting the first; hence $A^{(d^0)}$ reduces the binomial $A^{(d^1)}$ to one term.

The celebrated differential theorem of Laplace is only a particular case of the more general theorem (A). This will at once be seen, by making v equal to unity in equation (1).

---

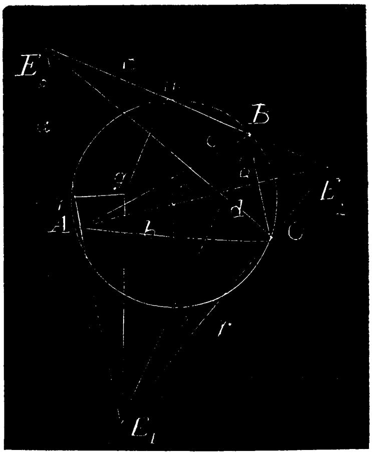
177. "If I_1, I_2, I_3 be the points of contact of the inscribed circle with the sides of the triangle ABC, E_1, E_2, E_3 the centres of the escribed circles, r_i the radius of the circle inscribed in $I_1I_2I_3$ and r_e the radius of that inscribed in $E_1E_2E_3$, show that

$$r_i=r\frac{a+b+c}{\alpha+\beta+\gamma}, \qquad r_e=2R\frac{a+b+c}{\alpha+\beta+\gamma},$$

where R and r have their usual values and α, β, γ are the distances between the centres of the escribed circles."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

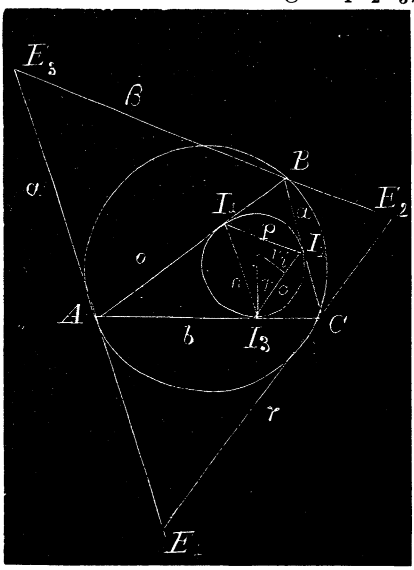
Let ABC be the given triangle, E_1, E_2, E_3 the centres of the escribed circles, d the centre of the inscribed and O the centre of the circumscribed circle.



Now it is well known that the lines E_3E_2, E_3E_1 and E_1E_2 are bisected by the circumference of the circle $ABCm$, or that the circle $ABCv$ is the nine point circle of $\triangle E_1E_2E_3$, and therefore that qE_1 or $R_e = 2R$; but this can be shown without the nine point circle. Because $E_1d = 2qm = 2E_1v$, and therefore qm is equal and parallel to E_1v. Therefore $E_1q = vm = 2R$. But E_1B, E_2A and E_3C are the perpendiculars of the triangle $E_1E_2E_3$, and by Ex. 226, page 320, Chauvenet's Geometry, $R_e\times(a+b+c) = 2\triangle E_1E_2E_3$, or $2R\times(a+b+c) = 2\triangle E_1E_2E_3$; and $r_e\times(a+\beta+\gamma) = 2\triangle E_1E_2E_3$; $\therefore$ $2R(a+b+c) = r_e(a+\beta+\gamma$, and hence

$$r_e = 2R.\frac{a+b+c}{a+\beta+\gamma}.$$

Again, let r_i be the radius of the inscribed circle of the triangle $I_1I_2I_3$, and r that of the triangle ABC which is the circumscribed circle of the triangle $I_1I_2I_3$. It is plain that E_3E_2 is parallel to I_1I_2, and therefore $I_1I_2I_3$, $E_1E_2E_3$ are similar, whence we get the following equations:



$R_e\times(o+p+n) = r\times(a+\beta+\gamma) = 2R(o+p+n)$. (1)

Also, $r_i(a+\beta+\gamma) = r_e(o+p+n)$. (2)

But from the preceding demonstration $r_e(a+\beta+\gamma) = 2R(a+b+c)$, and $2R(o+p+n) = r(a+\beta+\gamma)$, from (1).

Therefore by multiplying these two eq'ns we get $r_e(o+p+n) = r(a+b+c) = r_i(a+\beta+\gamma)$, from (2). Hence

$$r_i = r.\frac{a+b+c}{a+\beta+\gamma}.$$

Cor. $2R$, r_e, r and r_i are in geometrical proportion; also $\triangle E_1E_2E_3$ and its circumscribed circle, the circle ABC and the nine point circle of the triangle ABC are in geometrical proportion.

SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.

The sides of the triangle $I_1 I_2 I_3$ are $2r\cos\frac{1}{2}A$, &c. Half the sum of the sides is then $r(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C)$.

The radius of the circle circumscribed about $I_1 I_2 I_3$ is r. Substituting these values in the formula $r = abc\div 4Rs$ we have

$$r_i = \frac{8r^3\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{4r^2(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C)}$$

$$= \frac{rs}{2R(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C)}.$$

But it is known that $\alpha+\beta+\gamma = 4R(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C)$, hence

$$r_i = r\,.\frac{a+b+c}{\alpha+\beta+\gamma}.$$

It has been shown that the area of the triangle $E_1 E_2 E_3$ is $2Rs$; but the radius of the inscribed circle is equal to the area divided by half the perimeter, hence

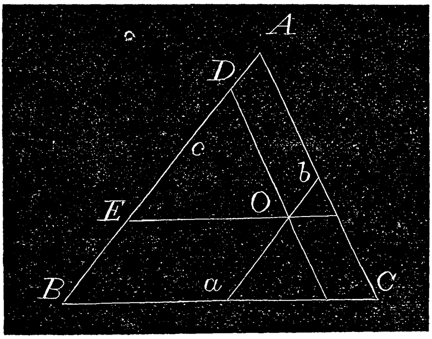
$$r_e = \frac{4Rs}{\alpha+\beta+\gamma} = 2R\frac{a+b+c}{\alpha+\beta+\gamma}.$$

---

178. "Through any point O in a plane triangle ABC three lines α, β, γ are drawn parallel to the sides a, b and c respectively; prove that

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c} = 2."$$

SOLUTION BY EDGAR FRISBY, NAVAL OBS., WASH., D. C.



$\frac{\alpha}{a} = \frac{AE}{c}$, by similar triangles; also

$\frac{\beta}{b} = \frac{BD}{c}$, " " " , and

$\frac{\gamma}{c} = \frac{BE+AD}{c}$, from opposite sides of parallelograms BO and AO.

By addition we have

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c} = \frac{(AE+BE)+(BD+AD)}{c}$$

$= 2c\div c = 2$.

---

179. "Integrate $\quad \dfrac{\sin\theta d\theta}{(\sin\theta+\cos\theta)^3}$."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

$$\int\frac{\sin\theta\,d\theta}{(\sin\theta+\cos\theta)^3}=\int\frac{d\theta}{(\sin\theta+\cos\theta)^2}-\int\frac{\cos\theta\,d\theta}{(\sin\theta+\cos\theta)^3}$$
$$=\int\frac{\sec^2\theta\,d\theta}{(1+\tan\theta)^2}-\int\frac{\sec^2\theta\,d\theta}{(1+\tan\theta)^3}$$
$$=\frac{1}{2(1+\tan\theta)^2}-\frac{1}{1+\tan\theta}+C.$$

SOLUTION BY GEO. LILLEY, KEWANEE, ILL.

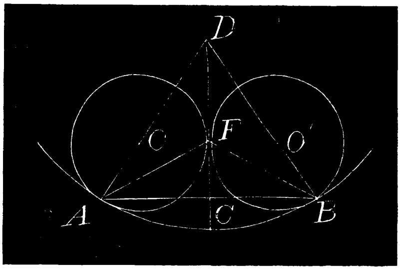
Let $u=\int\frac{\sin\theta\,d\theta}{(\sin\theta+\cos\theta)^3}=\int\frac{\tan\theta\sec^2\theta\,d\theta}{(\tan\theta+1)^3}$, and put $x=\tan\theta$; then
$$u=\int\frac{xdx}{(x+1)^3}.$$
Put $(x+1)^3=v$, then $u=\frac{1}{3}\int\frac{v^{\frac{1}{3}}-1}{\sqrt[3]{v^5}}dv=\frac{1-2v^{\frac{1}{3}}}{2\sqrt[3]{v^2}}=-\frac{1+2\tan\theta}{2(1+\tan\theta)^2}+C.$

---

180. "Two equal spheres placed in a paraboloid with its axis vertical touch one another at the focus. If W be the weight of a sphere, R, R' the pressures upon it, prove that $W^2:R.R'::3:2$."

SOLUTION BY HENRY HEATON, SABULA, IOWA.

Let O and O' be the centers of the two spheres, F the focus, CD the axis, and ACB a section of the surface of the paraboloid, the spheres being tangent to the surface at A and B.



AD and BD drawn through O and O' are normals to ACB. Put $AE=y$, $CE=x$ and ED, the subnormal, $=p$.

Then $y^2=2px$, and $FE=\frac{1}{2}p-x=(p^2-y^2)\div 2p$. $\angle DAE=2\angle FAE$. $\therefore$
$$\tan^{-1}\left(\frac{p}{y}\right)=2\tan^{-1}\left(\frac{p^2-y^2}{2py}\right)=\tan^{-1}\left(\frac{4py(p^2-y^2)}{-y^4+6p^2y^2-p^4}\right).$$
Therefore $\frac{p}{y}=\frac{4py(p^2-y^2)}{-y^4+6p^2y^2-p^4}$, and $p^2=3y^2$. $\therefore AD=2y$.

Since the pressures R and R' are in the directions of AD and AE, and the weight acts parallel to DE, $R:R':W::2y:y:y\sqrt{3}::2:1:\sqrt{3}$.

Therefore $\frac{1}{2}R=R'=W\sqrt{\frac{1}{3}}$; $\therefore\frac{1}{2}RR'=\frac{1}{3}W^2$; or $W^2:RR'::3:2$.

181 "Find the volume between $x = 0$ and $x = 2l$ of the solid bounded by the surface whose equation is

$$a(y^4+z^4) - z^2(y^3-2ay^2+2e^3) - y^2(bx^2+c^2x+e^3) = 0."$$

SOLUTION BY E. B. SEITZ.

Let $y = r\sin\theta$ and $z = r\cos\theta$. Then the equation becomes

$$ar^2 = (x^3+2e^3)\cos^2\theta+(bx^2+c^2x+e^3)\sin^2\theta,$$

and

$$V = \int\int\int r dr d\theta dx.$$

The limits of r are 0 and $\{[(x^3+2e^3)\cos^2\theta+(bx^2+c^2x+e^3)\sin^2\theta]\div a\}^{½}$ $= u$; of θ, 0 and 2π; and of x, 0 and $2l$.

$$\therefore\ V = \int_0^{2l}\int_0^{2\pi}\int_0^u r dr d\theta dx = \frac{1}{4a}\int_0^{2l}\int_0^{2\pi}\Big[x^3+bx^2+c^2x+3e^3+$$
$$(x^3-bx^2-c^2x+e^3)\cos 2\theta\Big]d\theta\, dx$$
$$= \frac{\pi}{2a}\int_0^{2l}(x^3+bx^2+c^2x+3e^3)dx = \frac{\pi l}{3a}\Big(6l^3+4bl^2+3c^2l+9e^3\Big).$$

NOTE BY THE EDITOR.—IN the *Note on Attraction*, at page 181, we modified Mr. Adcock's language, supposing that we had retained his ideas (except in one particular, which was thought to be a mistake), and sent him a copy of the "proof" for his approval or rejection. But, unfortunately, the proof was not returned until after the sheet was printed. Therefore, as Mr. Adcock does not accept the modification, we subjoin his note, *verbatim*.

"*Ultimate Proposition in Attraction.*— If the points of every particle of matter attract the points of every other particle of matter, then the resultant attraction between any two particles, whose dimensions are infinitely small in comparison with the distance between them, will be directly as the product of their masses and inversely as the square of the distance between them.

"That it is as the product of the masses results from the consideration that if the densities be increased or diminished in any ratio the number of attracting points or forces is increased or diminished in the same ratio without affecting the directions.

"That it is in the inverse ratio of the square of the distance follows, first, from the fact that any two points, of matter of the same density, attract each other with a constant force for all distances because the point of application of a force may be anywhere on its line of direction, distance being a quantity of a different kind has no effect on force; second, each point of